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THE EXISTENCE OF RESTRICTED RESOLVABLE DESIGNS I: (1, 2)-FACTORIZATIONS OF K_{2n}

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It has been known for at least a century that (in modern terminology) the complete graph K_{2n} admits a 1-factorization, that is, a partition of its edge set E into subsets $E_1, \ldots, E_{2n-1}$ such that each E_i consists of n vertex-disjoint edges. A considerably newer result (due to Ray-Chaudhuri and Wilson) states that if n is an odd integer then the complete graph K_{3n} admits what we will call a 2-factorization, that being a pair (T, P) where T is a decomposition of K_{3n} into triangles (K_3's) and P is a partition of T into subsets $T_1, \ldots, T_{(3n-1)/2}$ so that each T_i consists of n vertex-disjoint triangles. Between these two extremes we define a (1, 2)-*factorization of K_n with cardinality k* to be a pair (T, P) where T is a decomposition of K_n into edges and triangles (K_2's and K_3's) and P is a partition of T into subsets $T_1, \ldots, T_k$ such that each T_i is a vertex partition of K_n. This is the first in a series of two papers in which we investigate the following question: for which integers $n>0$ and $\lfloor n/2 \rfloor \leq k \leq n-1$ does the complete graph K_n admit a $(1,2)$-factorization of cardinality k? We prove here that when n is even the 'obvious' necessary conditions for the existence of these designs are sufficient, with exactly two exceptions: $n=6$, $k=3$; and $n=12$, $k=6$.

1. Introduction

Let G be a (finite) graph. A *decomposition* of G is a collection $\{G_1, \ldots, G_r\}$ of subgraphs of G with the property that each edge of G is contained in exactly one of the G_i. A *K-factor* in G is a subgraph $F \subseteq G$ containing all the vertices of G, so that each vertex has degree an integer from the set K. A *K-factorization* of G is a pair (D, P) where D is a decomposition of G and P is a partition of D into K-factors $F_1, \ldots, F_k$ (i.e. $\bigcup_{G_j \in F_i} G_j$ is a K-factor in G for each i). The integer $k = |P|$ is called the *cardinality* of the factorization.

We will be exclusively concerned with factorizations (of the complete graph K_n) in which each subgraph in the corresponding decomposition is complete, and henceforth the term 'factorization' will always mean one of this type.

A *pairwise balanced design* (PBD) is a pair (X, B) where X is a set of objects called *points* and B is a collection of subsets of X, called *blocks*, such that each pair of distinct points occurs in a unique block. A *parallel class* of blocks in a PBD is a sub-collection $B_1 \subseteq B$ that partitions the point set. A PBD is called *resolvable* if its block set can be partitioned (or *resolved*) into parallel classes $B_1, \ldots, B_k$. The integer k is called the *replication number* of the design. It is

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clear, then, that a PBD with v points is equivalent to a decomposition of K_v into complete subgraphs; a resolvable PBD (with replication number k) is equivalent to a factorization of K_v with cardinality k.

Factorizations of the complete graph have been historically connected with various scheduling problems. For example, the study of 1-factorizations (and their derivations, e.g. Room Squares, Balanced Tournament Designs, etc.) has evolved around the desire to construct various types of schedules for round-robin tournaments (see e.g. [9]). The first documented interest in 2-factorizations appeared in the form of Kirkman's famous Schoolgirl Problem (see [12]): fifteen schoolgirls are to be taken on walks, one a day over seven days, and on each walk are to be arranged into five groups of three each. Arrange a schedule so that each girl walks with each other girl in some group exactly once. Such a schedule is equivalent to a 2-factorization of K_{15}. A 2-factorization of K_v is usually called a *Kirkman Triple System*, or KTS(v). It was Ray-Chaudhuri and Wilson [12] who gave necessary and sufficient conditions for the existence of these designs.

Theorem 1.1. *There exists a Kirkman Triple System of order v if and only if $v \equiv 3$ modulo* 6.

A walking schedule for the schoolgirls as described above could, of course, just as easily be a schedule for a round-robin tournament with 15 players and 7 rounds provided that the game allows (at least) three players to compete simultaneously (e.g. darts or chinese checkers, but not hockey or chess). Suppose for example that there are 12 people who would like to arrange a round-robin dart tournament, but that they have enough time to play only seven rounds. We give two possible schedules (i.e. factorizations of K_{12} with cardinality 7) below.

Schedule I

1, 2, 3, 4	1, 5, 9	1, 7, 10	1, 8	3, 10	4, 11	8, 10
5, 6, 7, 8	2, 6, 10	2, 8, 11	2, 5	6, 12	7, 9	3, 6
9, 10, 11, 12	3, 7, 11	3, 5, 12	6, 11	8, 9	5, 10	1, 12
	4, 8, 12	4, 6, 9	3, 9	4, 5	1, 6	5, 11
			7, 12	1, 11	2, 12	4, 7
			4, 10	2, 7	3, 8	2, 9

Schedule II

1, 3	1, 2	1, 4	1, 5, 9	1, 6, 10	1, 7, 11	1, 8, 12
2, 4	3, 4	2, 3	3, 6, 12	2, 7, 12	2, 6, 9	2, 5, 10
5, 7	5, 6	5, 8	2, 8, 11	3, 5, 11	3, 8, 10	3, 7, 9
6, 8	7, 8	6, 7	4, 7, 10	4, 8, 9	4, 5, 12	4, 6, 11
9, 11	9, 10	9, 12				
10, 12	11, 12	10, 11				

There is another setting in which factorizations of complete graphs (i.e. resolvable designs) arise very naturally, and that is in the determination of $g^{(k)}(v)$, the smallest number of blocks required to construct a PBD on v points in which the largest block has size k. The following lower bound on $g^{(k)}(v)$ was given by Stinson [23].

Theorem 1.2. *If a* PBD *on v points has a block of size k then there are at least $1 + (v-k) \cdot (2rk - (v-k-1))/(r^2+r)$ blocks, where $r = \lfloor (v-1)/k \rfloor$. Equality is achieved if and only if each block has size $r+1$ or $r+2$ and intersects the block of size k.*

By removing the block of size k and all of its points, we see that a configuration achieving the bound of Theorem 1.2 is equivalent to a pairwise balanced design on $v-k$ points, with block sizes r and $r+1$, whose blocks can be resolved into k parallel classes. These latter designs have come to be known as *restricted resolvable designs,* and are denoted $R_rRP(v-k, k)$ (see [19]). Thus an $R_rRP(p, k)$ is equivalent to an $r-1$, r-factorization of K_p with cardinality k. In particular if $r=1$ a 0, 1-factorization of K_p is just a proper edge-colouring, and so exists when $k \geqslant p$, or $k = p-1$ and k is odd. In view of Theorem 1.2 this gives the following result, first proven by Pullman and Donald [11] and independently by Stanton, Allston and Cowan [21].

Theorem 1.3. *If $k+1 \leqslant v \leqslant 2k$ or $v = 2k+1$ where k is odd then $g^{(k)}(v) = 1 + \frac{1}{2}(v-k)(3k-v+1)$, and any optimal configuration consists of blocks of sizes 2 and 3 (together with the one block of size k), all of which intersect the block of size k.*

Let us now return briefly to the problem of constructing round-robin schedules. Consider the two schedules given for the 12-player 7-round tournament. In schedule I there are, in total, 35 games played (i.e. the design has 35 blocks) while in schedule II there are only 34 games played. Thus if the dart boards were being rented on a cost per game basis schedule II would be preferable to schedule I. More generally, suppose that we are given a schedule for a p-player k-round round-robin tournament (note that the mere existence of such a schedule presupposes that the game allow at least $1 + \lceil (p-1)/k \rceil$ players to compete simultaneously), i.e. a resolvable PBD on p points with replication number k. Let us assume that in this design no block contains all of the points (which in practice means that $k \neq 1$). If b is a block and x is a point not contained in b, then x is contained in $|b|$ blocks $c_1, \ldots, c_{|b|}$ each of which intersects b. Then $\{b, c_1, \ldots, c_{|b|}\}$ is a set of blocks in the design, no two of which can occur together in the same parallel class. Thus $k \geqslant |b|+1$, i.e. each block has size at most $k-1$. That means we can add k new points to our 'tournament design', each new point completing one of the parallel classes, to obtain a PBD on $p+k$

points in which the largest block has size k. Thus if N is the total number of games being played over the duration of the tournament we have $N \geqslant g^{(k)}(p+k)-1$ (note that we are counting a bye as a game played). Theorem 1.2 now implies the following.

Theorem 1.4. *Let $N(p, k)$ denote the smallest number of games able to be played in a p-player k-round round-robin tournament. Then*

$$N(p, k) \geqslant p \cdot \frac{2rk - p + 1}{r^2 + r}, \quad where \quad r = 1 + \left[\frac{p-1}{k}\right].$$

Equality can occur if and only if there exists an $R_rRP(p, k)$.

Thus for example schedule II (which is an $R_2RP(12, 7)$) yields the smallest number of games possible in a 12-player 7-round round-robin tournament.

The purpose of this series of papers is to investigate the existence of restricted resolvable designs $R_2RP(p, k)$, which we henceforth abbreviate as RRP(p, k). We also investigate the following question: suppose that we are given integers p and k for which an RRP(p, k) *cannot* exist (we will discuss necessary conditions later in the section). How 'close' can we come to such a design? By 'close' we will mean the following. It is noted in [15] that where $r = 2$ the bound of Theorem 1.2 can be amended to read:

Theorem 1.5. *Let $2k+1 \leqslant v \leqslant 3k+1$. Then $g^{(k)}(v) \geqslant 1 + \lceil (v-k) \cdot (4k - (v-k-1))/6 \rceil$ with equality occurring if and only if there exists a resolvable* PBD *on $v-k$ points, with replication number k, satisfying (i) each block has size* 1, 2, 3 *or* 4 *and (ii) there are at most two 'aberrant' blocks, i.e. of size* 1 *or* 4.

We will denote these more general designs by R*RP$(v-k, k)$. Note that an RRP(p, k) is (precisely) an R*RP(p, k) in which there are no aberrant blocks. The following designs are both R*RP(8, 5)'s.

I

1, 2, 3	4, 5, 6	7, 8	1, 5	1, 6, 8
4, 7	1, 7	1, 4	2, 6, 7	2, 4
5, 8	2, 8	2, 5	3, 4, 8	3, 5, 7
6	3	3, 6		

II

1, 2, 3, 4	1, 5	1, 6	1, 7	1, 8
5, 6, 7, 8	2, 6	2, 7	2, 8	2, 5
	3, 7	3, 8	3, 5	3, 6
	4, 8	4, 5	4, 6	4, 7

Both of these designs contain 18 blocks, which respresents the smallest number of games possible in an 8-player 5-round round-robin tournament. Of course two of the "games" in schedule I are not actually games at all but rather represent byes, so that one or the other of these schedules may be preferable in a given circumstance. In this paper we will be predominantly interested in R*RP's of the type in schedule I, i.e. in which the aberrant block(s) have size 1 (such designs could properly be referred to as *near*-(1, 2)-*factorizations of* K_p *with cardinality* k).

The existence of restricted resolvable designs RRP(p, k) (i.e. (1,2)-factoriztions of K_p with cardinality k) has been considered by many authors and for a variety of reasons. We have already indicated that Kirkman Triple Systems are known to exist for all orders congruent to 3 modulo 6. In [7], Kotzig and Rosa posed the problem of determining how 'close' one can come to a Kirkman Triple System when the number of points is a multiple of six. Specifically, can the complete graph K_{6n} be decomposed into one 1-factor and $3n-1$ 2-factors? Such a design is called a *Nearly Kirkman Triple system* NKTS$(6n)$, and the problem of determining the existence of these systems has only recently been completely settled (see [7, 1, 3 and 17]).

Theorem 1.6. *There exists a Nearly Kirkman Triple System* NKTS$(6n)$ *if and only if* $n \geqslant 3$.

Rees and Wallis [19] gave constructions to prove the following.

Theorem 1.7. *Let* $p \equiv 0 \mod 6$. *There exist* RRP(p, k) *for all* $p/2 \leqslant k \leqslant p-1$, *with the exceptions* $(p, k) = (6, 3)$ *or* $(12, 6)$.

When $p \equiv 0 \mod 6$ (and $p/2 \leqslant k \leqslant p-1$) a question that arises naturally is the following: can one construct an RRP(p, k) in which each parallel class consists either entirely of blocks of size 2 or entirely of blocks of size 3? (The designs constructed in [19] do not in general have this property.) Equivalently, given non-negative integers a, b and n satisfying

$$a + 2b = 6n - 1 \tag{1.1}$$

can complete graph K_{6n} be decomposed into a 1-factors and b 2-factors? Thus, for example, an NKTS$(6n)$ corresponds to the case $a = 1$. This problem was solved completely by the author in [14]; the only solutions (a, b, n) to Eq. (1.1) for which such a decomposition of K_{6n} does *not* exist are those corresponding to the non-existent Nearly Kirkman Triple Systems, i.e. $(a, b, n) = (1, 2, 1)$ or $(1, 5, 2)$.

These designs admit to an interesting algebraic interpretation. Thus suppose that we are given a decomposition of K_{6n} into a 1-factors and b 2-factors. Label the vertices $0, 1, \ldots, 6n-1$, and assign a specific orientation to each triangle in the design (i.e. each triangle becomes a directed 3-cycle). Define $0 * j = j$ for all j, and if $i \neq 0$ proceed as follows:

(i) If $\{0, i\}$ belongs to a one-factor F in the design we define $i * j = k$ where $\{j, k\}$ belongs to F,

(ii) If $\{0, i\}$ belongs to the directed 3-cycle $0, i, l$ in the design define $i * j = k$ where j, k, m is the directed 3-cycle belonging to the same 2-factor as does $0, i, l$,

(iii) If $\{0, i\}$ belongs to the directed 3-cycle $0, l, i$ in the design define $i * j = k$ where j, m, k is the directed 3-cycle belonging to the same 2-factor as does $0, l, i$.

The operation $*$ defines a quasigroup with an identity element (note that $0 * j = j * 0 = j$ for all j), i.e. a loop. Additionally,

(i) each element $i \neq 0$ generates a group of order 2 or 3 (there are a elements of order 2 and $2b$ elements of order 3), and

(ii) the identity $i^a * (i^b * j) = i^{a+b} * j$ holds for all i, j.

Conversely, any loop with v elements satisfying properties (i) and (ii) can be used to generate a decomposition of K_v into a 1-factors and b 2-factors (the factors are just the cosets of cyclic subgroups).

It is obvious that if an RRP(p, k) exists $\lfloor p/2 \rfloor \leq k \leq p-1$. Moreover, each point is contained in $p-1-k$ triples and $2k-p+1$ pairs so that the number of blocks in the design is $p \cdot (\frac{1}{3}(p-1-k) + \frac{1}{2}(2k-p+1)) = \frac{1}{6}p(4k-p+1)$. Thus we must have $p \cdot (k-p+1) \equiv 0 \mod 3$. In this paper we will give constructions to prove the following (see Section 3).

Theorem 1.8. *Let p be an even integer with $p \equiv 2$ or $4 \mod 6$. There exists a restricted resolvable design* RRP(p, k) *if and only if $p/2 \leq k \leq p-1$ and $p \cdot (k-p+1) \equiv 0 \mod 3$.*

Together with Theorem 1.7 this will complete the spectrum for RRP$(2n, k)$.

Regarding the more general class of designs R*RP(p, k) the parameter k must (in general) satisfy $\lfloor p/2 \rfloor \leq k \leq p-1$. There are a few trivial exceptions, but this relation certainly must hold when $p \geq 9$ since there can be at most two aberrant blocks, which between them cannot cover more than 8 points. Where p is even there is one "troublesome" case, as is illustrated by the following lemma:

Lemma 1.9. *Let $p \equiv 4 \mod 6$. There exists an* R*RP$(p, p/2)$ *if and only if $p = 10$ and possibly $p = 16$.*

Proof. Suppose that we have an R*RP$(p, p/2)$. Let b_i denote the number of blocks of size i in the design, and for each point x let x_i denote the number of blocks of size i containing x, where $i = 1, 2, 3, 4$. Then $3x_4 + 2x_3 + x_2 = p-1$ and $x_1 + x_2 + x_3 + x_4 = p/2$, whence $x_4 + 1 = x_2 + 2x_1$. Summing this last equation over the points in the design and recalling that $b_4 \leq 2$ we have $b_2 \leq \frac{1}{2}(p+8)$. On the other hand since $b_1 + b_4 \leq 2$ we deduce that at least $p/2 - 2$ parallel classes contain only pairs and triples; since $p \equiv 4 \pmod 6$ this implies $b_2 \geqslant 2((p/2) - 2) = p - 4$. Thus $p - 4 \leqslant \frac{1}{2}(p + 8)$, or $p \leqslant 16$. It is easy to see that there is no R*RP(4, 2); on the other hand we do have an R*RP(10, 5): take a resolvable TD(3, 4) and remove two points from one of the groups—now just identify groups as blocks. We do not know if there is an R*RP(16, 8). □

In Section 4 we will show that given any even integer $p \geqslant 96$ and any k with $p/2 \leqslant k \leqslant p - 1$ (where $k \neq p/2$ when $p \equiv 4 \pmod 6$) there exists an R*RP(p, k) with block sizes from $\{1, 2, 3\}$, i.e. a near-(1, 2)-factorization of K_p with cardinality k.

2. Preliminaries – techniques for construction

In this section we will indicate the various techniques which we shall use to construct our designs. The single most important tool will be a class of combinatorial designs called frames, and in particular we will make essential use of the results in [16]. We now review some definitions. A *group divisible design* (GDD) is a triple (X, G, B) where X is a set of *points*, G is a partition of X into subsets called *groups*, and B is a collection of subsets of X called *blocks* such that any pair of distinct points occurs in either a unique group or a unique block, but not in both. A *transversal design* TD(k, n) is a group divisible design on nk points with k groups of size n in which every block has size k. It is well known that a TD(k, n) coexists with a set of $k - 2$ mutually orthogonal latin squares of order n. In particular there exist TD(3, n) for all $n > 0$ and TD(4, n) for all $n > 0$ except $n = 2, 6$. A GDD is called *resolvable* if its block set can be partitioned into parallel classes. The replication number of a resolvable GDD is the number of parallel classes contained in any such resolution of its blocks. A K-GDD of type $g_1^{t_1} g_2^{t_2} \cdots g_r^{t_r}$ is a GDD in which each block has size from K and in which there are t_i groups of size g_i, $i = 1, \ldots, r$ (we also say K-GDD of type S, where S is the multiset consisting of t_i copies of g_i, $(i = 1, \ldots, r)$. Note that in a resolvable k-GDD all the groups have the same size. In [17], Rees and Stinson investigated the existence of resolvable 3-GDD's and obtained the following result.

Theorem 2.1. *Let g and u be given with $gu \equiv 0 \pmod 3$ and $g(u - 1) \equiv 0 \pmod 2$, $(g, u) \neq (2, 3)$, $(2, 6)$ or $(6, 3)$. There exists a resolvable* 3-GDD *of type g^u, except possibly where $g \equiv 6 \pmod{12}$ and $u = 11, 14$ or $g \equiv 2$ or $10 \pmod{12}$ and $u = 6$.*

Constrictions for resolvable 3-GDD's of type g^6 where $g \equiv 0 \pmod 4$ were also given by Mendelsohn and Hao [8]. We will make use of Theorem 2.1, principally in connection with the following simple construction.

Construction 2.2. *Suppose that there is a resolvable* 2, 3-GDD *of type* S, *with replication number* k, *and that for each* $s_j \in S$ *there is an* RRP(s_j, k'). *Then there is an* RRP$(\sum_j s_j, k + k')$.

A *frame* is a group divisible design (X, G, B) whose block set can be partitioned into *holey* parallel classes, i.e. each holey parallel class is a partition of $X - G_i$ for some group G_i. The groups in a frame are referred to as *holes*. The *degree* of a hole $G_i \in G$ is the number of holey parallel classes that partition $X - G_i$. The following result is proven in [16].

Theorem 2.3. *Let* (X, G, B) *be a* 4, 5-GDD *of type* $g_1^{t_1} g_2^{t_2} \cdots g_r^{t_r}$, $G = \{G_i\}$, *and let* $\{d_1, \ldots, d_{|G|}\}$ *be any sequence of integers with* $3|G_i| \leq d_i \leq 6|G_i|$ *where* $d_i = 3|G_i| + 1$ *for at most one value of* i. *There exists a* 2, 3*-frame of type* $(6g_1)^{t_1}(6g_2)^{t_2} \cdots (6g_r)^{t_r}$ *in which the ith hole has degree* d_i, $i = 1, \ldots, |G|$.

Before indicating the main recursive construction for restricted resolvable designs from frames we need the notion of an RRP "missing" a sub-design. By an RRP(p, k) − RRP(w, d) (where $0 \leq w \leq p$ and $0 \leq d \leq k$) we mean a triple (X, G, B) where $(X, B \cup \{G\})$ is a pairwise balanced design on p points satisfying

(i) $|G| = w$ and $|b| = 2$ or 3 for each $b \in B$, and

(ii) B admits a partition into subsets $B_1, \ldots, B_k$ where for each $i = 1, \ldots, d$ B_i is a partition of $X - G$, and for each $i = d + 1, \ldots, k$ B_i is a partition of X.

(The above design is referred to in [15] as a resolvable PBD$(\{2, 3\}, k - d; p)$ with a hole of size w and degree d.) Note that if there exists also an RRP(w, d) we can build this latter design on the points of G to obtain an RRP(p, k) with a sub-RRP(w, d).

Construction 2.4. *Let* (X, G, B) *be a* 2, 3*-frame in which hole* G_i *has degree* d_i, $i = 1, \ldots, j$. *Suppose that for each* $i = 1, \ldots, j - 1$ *there is an* RRP$(|G_i| + w, d_i + d)$ − RRP(w, d) *and that there is an* RRP$(|G_j| + w, d_j + d)$. *Then there is an* RRP$(w + \sum |G_i|, d + \sum d_i)$.

This construction works by adding w 'ideal' points to the frame. For each $i = 1, \ldots, j - 1$ build the indicated design on the points of G_i together with the ideal points, in such a way that the 'missing' subdesign occurs on these w ideal points. Then on G_j together with the ideal points, we build an RRP$(|G_j| + w, d_j + d)$. The parallel classes in the finished design are obtained by 'pairing off'.

Construction 2.4 is equally applicable to constructing R*RP's. Define R*RP(p, k) − RRP(w, d) to be a triple (X, G, B) as above except that B may contain at most two blocks of size 1 or 4. Then we may use Construction 2.4, filling *one* of the holes in the frame with an R*RP$(|G_i| + w, d_i + d)$ − RRP(w, d) (or an R*RP$(|G_j| + w, d_j + d)$), and the resulting design is an R*RP.

Having Constructions 2.2 and 2.4 we now need a way to construct some 'small' restricted resolvable designs. For this we use the standard 'difference method' technique (see e.g. [4]), employing simple backtracking and/or hill-climbing algorithms to generate base blocks (by hand).

It will be useful in many instances to reduce a given problem into subproblems. Thus suppose that we wish to construct a $(1, 2)$-factorization of K_p with cardinality k. We could do this if we know that for some k_1, k_2 with $k_1 + k_2 = k$ there exist edge disjoint spanning subgraphs $H_1, H_2 \subseteq K_p$ with $H_1 \cup H_2 = K_p$, where H_i admits a $(1, 2)$-factorization with cardinality k_i, $i = 1, 2$. To this end we will find the following very useful. A graph G with v vertices is called *cyclic* if it admits $\mathbb{Z}_v$ as a vertex-transitive group of automorphisms. Thus one may label the vertices of a cyclic graph with the elements of $\mathbb{Z}_v$ in such a way that the vertex-to-vertex map $x \to x + i \pmod{v}$ is an automorphism of the graph for each $i \in \mathbb{Z}_v$. The *order* of an edge $\{x, y\}$ in a cyclic graph is the (additive) order of $x - y$ in $\mathbb{Z}_v$. The following result is due to Stern and Lenz [22].

Theorem 2.5. *Let G be a cyclic graph on v vertices having an edge (e.g. $\{0, v/2\}$) of even order. Then G has a 1-factorization.*

Corollary 2.6. *Let H be a cyclic graph on n vertices and let G be the graph obtained by taking two (disjoint) copies of H, written on the vertex sets $\{0_1, 1_1, \ldots, (n-1)_1\}$ and $\{0_2, 1_2, \ldots, (n-1)_2\}$, together with the edges $\{(x_1, (x+i)_2) : x \in \mathbb{Z}_n, i \in I\}$ where I is a non-empty subset of $\mathbb{Z}_n$. Then G has a 1-factorization.*

Proof. We may assume that I is the singleton set $\{i\}$, since for each $k \in \mathbb{Z}_n$ the edges $\{(x_1, (x+k)_2) : x \in \mathbb{Z}_n\}$ form a 1-factor in G.

Suppose that n is odd. We relable the vertices of G with the elements of $\mathbb{Z}_{2n}$, as follows. For each $0 \leq j \leq n-1$ relable j_1 by $2j \pmod{2n}$ and j_2 by $2(j-i) + n \pmod{2n}$. It is easily checked that the map $x \to x + 1 \pmod{2n}$ is an automorphism of G, and that G has an edge, namely $\{0, n\}$, of even order. By Theorem 2.5 G has a 1-factorization.

If n is even and H has an edge of even order, we are done by Theorem 2.5. If H has no edges of even order, and we let 2^k be the highest power of two dividing n, we can write G as the disconnected union $G = G_1 \cup G_2 \cup \cdots \cup G_{2^k}$ where $G_i \simeq G_j$ for all i, j and G_1 satisfies the hypothesis of the corollary. Since G_1 has $n/2^{k-1}$ vertices (twice an odd number) we can now proceed as above (i.e. the n odd case). □

We shall need the following results, the first of which appears as Theorems 3.3 and 3.5 in [19].

Theorem 2.7. *Let $n \geq 5$ be odd. There exist* RRP$(3n, k)$ *for $k = 2n - 2$, $2n - 1$, $2n$ and $2n + 1$. Also, there exist* RRP$(9, 4)$ *and* RRP$(9, 5)$.

Lemma 2.8. *There exist* RRP(21, 16) *and* RRP(27, 22).

Proof. Take the point set $\{a_1, \ldots, a_9\} \cup \mathbb{Z}_{12}$. The blocks

$$\begin{array}{llllll} a_1 2 & a_6 8 & & & & \\ a_2 4 & a_7 9 & & & & \\ a_3 5 & a_8 10; & 0, 4, 8; & 0, 5; & 0, 6 & \bmod 12 \\ a_4 6 & a_9 11 & & & & \\ a_5 7 & 0, 1, 3 & & & & \end{array}$$

yield an RRP(21, 16) − RRP(9, 4). Building a KTS(9) on the points $\{a_1, \ldots, a_9\}$ gives an RRP(21, 16). To get an RRP(27, 22) we proceed similarly, with point set $\{a_1, \ldots, a_9\} \cup \mathbb{Z}_{18}$, starting with the blocks

$$\begin{array}{lllllll} a_1 0 & a_5 6 & a_9 16 & & & & \\ a_2 2 & a_6 8 & 1, 9 & & & & \\ a_3 4 & a_7 11 & 3, 7; & 0, 6, 12; & 0, 5; & 0, 9 & \bmod 18. \\ a_4 5 & a_8 14 & 10, 17 & & & & \\ & & 12, 13, 15 & & & & \end{array}$$

(Note that in each case the set of blocks $0, 5 \mod m$ ($m = 12, 18$) yields two 1-factors.) □

3. The existence results for RRP's

In this section we obtain a proof of Theorem 1.8. Let $\mathcal{R} = \{p \in \mathbb{Z}^+;$ there exists an RRP(p, k) for all $\lfloor p/2 \rfloor \leqslant k \leqslant p-1$ with $p(k-p+1) \equiv 0 \mod 3\}$. Thus we wish to show that $\{p \in \mathbb{Z}^+ : p \equiv 2$ or $4 \mod 6\} \subseteq \mathcal{R}$. We do this principally by means of the following lemma.

Lemma 3.1. *Suppose that there exists a* 4, 5-GDD *of type* $3^{t_1}4^{t_2}r^1$ *where* $t_1 + t_2 > 0$ *and* r *is any non-negative integer, and let* $s = 3t_1 + 4t_2 + r$ (*i.e.* s *is the number of points in the* GDD). *Then* $6s + 2 \in \mathcal{R}$ *and* $6s + 4 \in \mathcal{R}$.

Proof. Suppose first that $p = 6s + 2$ and let k be given, with $p/2 \leqslant k \leqslant p-1$ and $p(k-p+1) \equiv 0 \mod 3$. Let $q = \frac{1}{3}(k - p/2)$ and let $n_1, \ldots, n_{t_1+t_2}, n_\infty$ be any sequence of integers with $0 \leqslant n_i \leqslant 3$ for $i = 1, \ldots, t_1$, $0 \leqslant n_i \leqslant 4$ for $i = t_1 + 1, \ldots, t_1 + t_2$ and $n_\infty \in \{0, r\}$, satisfying $n_\infty + \sum n_i = q$. That such a sequence exists is a consequence of the following observations:

(i) $0 \leqslant q \leqslant \frac{1}{3}(p - 1 - (p/2)) = (p-2)/6 = s$, and

(ii) since $t_1 + t_2 > 0$ the existence of a 4, 5-GDD of type $3^{t_1}4^{t_2}r^1$ clearly implies that $3t_1 + 4t_2 \geqslant 2r + 3 > r$.

Now apply Theorem 2.3 to construct a 2, 3-frame of type $18^{t_1}24^{t_2}(6r)^1$ in which the ith hole H_i has degree d_i, where $d_i = 9 + 3n_i$ for $i = 1, \ldots, t_1$ and $d_i = 12 + 3n_i$

for $i = t_1 + 1, \ldots, t_1 + t_2$, and in which the hole H_∞ of size $6r$ has degree $3r + 3n_\infty$. Add two "ideal" points to this frame.

Use Construction 2.4, with $w = 2$ and $d = 1$. We have the required 'input' designs. An RRP($6r + 2, 3r + 3n_\infty + 1$) is either a 1-factorization of K_{6r+2} or, when $n_\infty = 0$, is obtained by removing a point from a Kirkman Triple System KTS($6r + 3$). All of the possible 20-point and 26-point RRP's 'missing' a sub-RRP(2, 1) are given in the appendix. We obtain a design with $18t_1 + 24t_2 + 6r + 2 = p$ points, with replication number

$$1 + 3r + 3n_\infty + \sum_{i=1}^{t_1} (9 + 3n_i) + \sum_{i=t_1+1}^{t_1+t_2} (12 + 3n_i) = 1 + \frac{p-2}{2} + 3q = k$$

as desired. Thus $p = 6s + 2 \in \mathcal{R}$.

To show that $p = 6s + 4 \in \mathcal{R}$ we proceed in similar fashion, letting $q = \frac{1}{3}(k - ((p/2) + 1))$, and adding four 'ideal' points to the resulting frame. Use Construction 2.4 with $w = 4$ and $d = 3$. Regarding the 'input' designs, an RRP($6r + 4, 6r + 3$) is a 1-factorization of K_{6r+4}; an RRP($6r + 4, 3r + 3$) is either a 1-factorization of K_4, an RRP(10, 6) (see later in this section) or can be obtained by removing a block of size two from a Nearly Kirkman Triple System NKTS($6r + 6$). All of the possible 22-point and 28-point RRP's 'missing' a sub-RRP(4, 3) are given in the appendix.

This completes the proof of Lemma 3.1. □

We now need some group divisible designs. The following result is a consequence of the work of Hanani, Ray-Chaudhuri and Wilson concerning the existence of resolvable designs with block size four [6].

Lemma 3.2 [16, Corollary 3.2]. *Let $t \equiv 1$ modulo 3, $t \geq 4$ and $0 \leq r \leq 4(t - 1)/3$. Then there exists a 4, 5-GDD of type $4^t r^1$.*

We can now prove:

Theorem 3.3. *Let $p = 6s + 2$ or $6s + 4$, where $s \notin \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 13, 14\}$. Then p is in the set $\mathcal{R}$.*

Proof. We apply Lemma 3.1. If $s \notin \{12, 15, 21, 22, 23, 24, 25, 26, 27, 37, 38, 39\}$ we can write $s = 4t + r$ where $t \equiv 1$ modulo 3, $t \geq 4$ and $0 \leq r \leq 4(t - 1)/3$ (e.g. let r be the least residue of $s - 4$ modulo 12); now use Lemma 3.2 to construct the relevant GDD on s points. The remaining values of s are settled as follows:

(i) $s = \mathbf{12, 15}$. Remove a point from the projective plane of order 3 or affine plane of order 4 to obtain 4-GDD's of types 3^4, 3^5;

(ii) $s = \mathbf{21, 22, 23, 24}$. Remove the appropriate number of points from a fixed block in the affine plane of order 5 to obtain 4, 5-GDD's of types $4^5 r^1$ (where $r = 1, 2, 3$ or 4);

(iii) $s = \mathbf{25, 26, 27}$. Bennett [2] has constructed a resolvable 4-GDD of type 3^8

(we give it in the appendix). By viewing this as a resolvable 3, 4-GDD of type 4^6 and adding a 'group at infinity' of the appropriate size we can thus construct 4, 5-GDD's of types 4^6r^1 (where $r = 1$, 2 or 3);

(vi) $s = \mathbf{37}, \mathbf{38}, \mathbf{39}$. Remove the appropriate number of points from a fixed block in a balanced incomplete block design (41, 5, 1)-BIBD (see [5]) to obtain 4, 5-GDD's of type 4^9r^1 (where $r = 1$, 2 or 3).

This completes the proof. □

To prove Theorem 1.7 we now need only show that $6s + 2$ and $6s + 4$ are in the set $\mathcal{R}$, where $s \in \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 13, 14\}$.

Our constructions will (for the most part) fall into two categories. For the 'low' replication numbers (i.e. k 'close' to $p/2$) we will use Construction 2.2, starting with resolvable transversal designs with block size 3. For higher values of k we will apply the Stern–Lenz hypothesis (via Corollary 2.6) in the following manner: writing $p = 2n$, we will use direct methods to construct n parallel classes in such a way that the remaining pairs form a $(k - n)$-regular graph on $2n$ vertices satisfying the hypothesis of Corollary 2.6. Constructing a 1-factorization of this graph yields the required design. In constructing the initial n parallel classes we will take $\mathbb{Z}_n \times \mathbb{Z}_2$ as the point set, building a parallel class B of base blocks which will then be developed mod $(n, -)$. To insure that the remaining pairs form a graph with the desired properties we will choose B so that

(i) at least one mixed difference does not occur in any block of B, and

(ii) if $d \in \mathbb{Z}_n$ occurs (in B) as a pure $(\mathbb{Z}_n \times \{0\})$-difference then d also occurs (in B) as a pure $(\mathbb{Z}_n \times \{1\})$-difference, and vice versa.

We will find the following observations useful.

(01) Suppose that B contains the blocks $a_0b_1c_1$ and $a_1d_0e_0$ and that 0 does not occur as a mixed difference in B. Then the set $(B - \{a_0b_1c_1, a_1d_0e_0\}) \cup \{a_0a_1, d_0e_0, b_1c_1\}$ satisfies properties (i) and (ii) above, and so determines an RRP$(p, k + 3)$.

(02) Suppose that B contains the blocks $a_0b_1c_1$, $a_1b_0c_0$, $d_0e_1f_1$ and $d_1e_0f_0$, where $a - d \not\equiv n/2$ when n is even, and that $\pm(a - d)$ does not occur as a pure difference in B. Then the set $(B - \{a_0b_1c_1, a_1b_0c_0, d_0e_1f_1, d_1e_0f_0\}) \cup \{a_0d_0, a_1d_1, b_0c_0, b_1c_1, e_0f_0, e_1f_1\}$ satisfies properties (i) and (ii) and so determines an RRP$(p, k + 6)$.

(03) Suppose that B contains the blocks $a_0(a + 1)_0(a + 3)_0$, $a_1(a + 1)_1(a + 3)_1$, $(a + 2)_0b_1c_1$ and $(a + 2)_1b_0c_0$. Then the set $(B - \{a_0(a + 1)_0(a + 3)_0, a_1(a + 1)_1(a + 3)_1, (a + 2)_0b_1c_1, (a + 2)_1b_0c_0\}) \cup \{a_0(a + 3)_0, a_1(a + 3)_1, (a + 1)_0(a + 2)_0, (a + 1)_1(a + 2)_1, b_0c_0, b_1c_1\}$ satisfies properties (i) and (ii) and so determines an RRP$(p, k + 6)$.

Finally, it will be convenient to construct B in such a way that the above observations can be applied repeatedly (thus constructing several RRP's from a single one). To this end we will arrange B so that its pure differences occur in the *odd starter* $(1, n - 1), (2, n - 2), \ldots, ((n - 1)/2, (n + 1)/2)$ if n is odd, or the *even starter* $(1, 2s-1), (2, 2s-2), \ldots, (s-1, s+1)$, $(2s, n-1), (2s+1, n-2), \ldots, (\lceil 3n/4 \rceil - 1, \lceil 3n/4 \rceil)$ (where $s = \lceil n/4 \rceil$) if n is even; where B contains pure triples (i.e. as in (03)) its pure differences will occur in the "refined" (even) starter $(1, 2s-1)$, $(2, 2s-2), \ldots, (s-2, s+2)$, $(2s, n-1)$, $(2s+1, n-2), \ldots, (\lceil 3n/4 \rceil - 3, \lceil 3n/4 \rceil + 2)$, $(\lceil 3n/4 \rceil - 2, \lceil 3n/4 \rceil - 1, \lceil 3n/4 \rceil + 1)$. In particular then if $a_0 b_1 c_1$ is a triple in B then (in general) so is $a_1 b_0 c_0$, $d_0 e_1 f_1$ and $d_1 e_0 f_0$ where (a, d) is in the relevant starter above. This will facilitate the repeated use of (02).

For the sake of brevity we will, for some values of n, omit any reference to the designs RRP$(2n, 2n-1)$. This is alright, of course, since these designs are just 1-factorizations.

RRP(2, 1), RRP(4, 3). These are 1-factorizations.

RRP(8, 4). Remove a point from a KTS(9).

RRP(8, 7). Take a 1-factorization of K_8.

RRP(10, 6).
$$\begin{array}{llll} 0_0 1_1 4_1 & 0_1 2_0 3_0 & & \\ 2_1 3_1 & 1_0 4_0 & ; \; 0_0 0_1 & \bmod (5, -) \end{array}$$

RRP(10, 9). Take a 1-factorization of K_{10}.

RRP(14, 7). Remove a point from a KTS(15).

RRP(14, 10)
$$\begin{array}{llll} 0_0 1_1 6_1 & 0_1 2_0 5_0 & & \\ 2_1 5_1 & 1_0 6_0 & ; \; 0_0 0_1; \; 0_0 3_1; \; 0_0 4_1 & \bmod (7, -) \\ 3_1 4_1 & 3_0 4_0 & & \end{array}$$

RRP(14, 13). Take a 1-factorization of K_{14}.

RRP(16, 9). Remove a block of size two from an NKTS(18).

RRP(16, 12)
$$\begin{array}{llll} 0_0 1_1 3_1 & 0_1 1_0 3_0 & & \\ 4_1 7_1 & 4_0 7_0 & ; \; 0_0 4_0; \; 0_0 2_1; \; 0_0 6_1; \; 0_0 4_1 & \bmod (8, -) \\ 5_1 6_1 & 5_0 6_0 & 0_1 4_1 & \\ & 2_0 2_1 & & \end{array}$$

RRP(16, 15). Take a 1-factorization of K_{16}.

RRP(20, k), $10 \leq k \leq 19$. See appendix.

RRP(22, k), $12 \leq k \leq 21$. See appendix (construct an RRP(4, 3) on the points $\{a, b, c, d\}$ in each design).

RRP(26, k), $13 \leq k \leq 25$. See appendix.

RRP(28, k), $15 \leq k \leq 27$. See appendix (construct an RRP(4, 3) on the points $\{a, b, c, d\}$ in each design, except where $k = 24$ in which case the relevant points are $\{a_1, a_4, 0, 9\}$).

RRP(32, 16). Remove a point from a KTS(33).

RRP(32, 19). Take a resolvable TD(3, 12) and remove four points from one of its groups to obtain a resolvable 2, 3-GDD of type $12^2 8^1$ with replication number 12. Apply Construction 2.2, filling in RRP(12, 7) (Theorem 1.6) and RRP(8, 7).

RRP(32, 22).

$$\text{Sixteen parallel classes:}\quad \begin{array}{ccc} 0_0 4_1 & & 0_1 4_0 \\ 1_0 7_0 & & 1_1 7_1 \\ 5_1 2_0 6_0 & & 5_0 2_1 6_1 \\ 3_0 9_1 14_1 & & 3_1 9_0 14_0 \\ 8_0 15_0 & & 8_1 15_1 \\ 10_0 11_0 13_0 & & 10_1 11_1 13_1 \\ & 12_0 12_1 & \end{array} \quad \mathrm{mod}\,(16, -)$$

Then apply Corollary 2.6.

RRP(32, 25).

$$\text{Sixteen parallel classes:}\quad \begin{array}{cc} 0_0 4_1 & 0_1 4_0 \\ 1_0 7_0 & 1_1 7_1 \\ 5_1 2_0 6_0 & 5_0 2_1 6_1 \\ 3_0 9_1 14_1 & 3_1 9_0 14_0 \\ 8_0 15_0 & 8_1 15_1 \\ 10_0 13_0 & 10_1 13_1 \\ 11_0 12_0 & 11_1 12_1 \end{array} \quad \mathrm{mod}\,(16, -)$$

Then apply Corollary 2.6.

RRP(32, 28), RRP(32, 31). Apply observations (01) and (02) to the above design (e.g. for $k = 28$, apply (01) to $5_1 2_0 6_0$ and $5_0 2_1 6_1$; for $k = 31$, apply (02) to $5_1 2_0 6_0$, $5_0 2_1 6_1$, $3_0 9_1 14_1$ and $3_1 9_0 14_0$.)

RRP(34, 18). Remove a block of size two from an NKTS(36).

RRP(34, 21). Take a resolvable TD(3, 12) and remove two points from one of the groups to obtain a resolvable 2, 3-GDD of type $12^2 10^1$ with replication number 12. Apply Construction 2.2, filling in RRP(12, 9) (Theorem 1.6) and RRP(10, 9).

RRP(34, 24).

$$\text{Seventeen parallel classes:}\quad \begin{array}{cc} 1_0 2_1 15_1 & 1_1 2_0 15_0 \\ 3_0 14_0 16_1 & 3_1 14_1 16_0 \\ 0_0 5_1 12_1 & 0_1 6_0 11_0 \\ 6_1 11_1 & 5_0 12_0 \\ 4_0 13_0 & 4_1 13_1 \\ 7_0 10_0 & 7_1 10_1 \\ 8_0 9_0 & 8_1 9_1 \end{array} \quad \mathrm{mod}\,(17, -)$$

Then apply Corollary 2.6.

RRP(34, k), 27 ≤ k ≤ 33. Apply observations (01) and (02) to the above design.

RRP(38, 19). Remove a point from a KTS(39).

RRP(38, 22). Take a resolvable TD(3, 15) and remove seven points from one of the groups to obtain a resolvable 2, 3-GDD of type $15^2 8^1$ with replaction number 15. Apply Construction 2.2, filling in RRP(15, 7) (i.e. KTS(15)) and RRP(8, 7).

RRP(38, 25).

Nineteen parallel classes:

$$\begin{array}{ccc} 4_0 15_0 & & 6_1 13_1 \\ 0_0 4_1 15_1 & & 6_0 13_0 0_1 \\ 1_0 18_1 & & 1_1 18_0 \\ 2_0 3_1 16_1 & & 2_1 3_0 16_0 \\ 5_0 14_0 17_1 & & 5_1 14_1 17_0 \\ 7_0 12_0 & & 7_1 12_1 \\ 8_0 9_0 11_0 & & 8_1 9_1 11_1 \\ & 10_0 10_1 & \end{array} \quad \text{mod } (19, -)$$

Then apply Corollary 2.6.

RRP(38, 28).

Nineteen parallel classes:

$$\begin{array}{cc} 1_0 2_1 17_1 & 1_1 2_0 17_0 \\ 18_0 3_1 16_1 & 18_1 3_0 16_0 \\ 0_0 5_1 14_1 & 0_1 6_0 13_0 \\ 6_1 13_1 & 5_0 14_0 \\ 4_0 15_0 & 4_1 15_1 \\ 7_0 12_0 & 7_1 12_1 \\ 8_0 11_0 & 8_1 11_1 \\ 9_0 10_0 & 9_1 10_1 \end{array} \quad \text{mod } (19, -)$$

Then apply Corollary 2.6.

RRP(38, k), $31 \leqslant k \leqslant 37$. Apply observations (01) and (02) to the above design.

RRP(40, 21), RRP(40, 24). Take a resolvable TD(3, 14) and remove two points from one of the groups to obtain a resolvable 2, 3-GDD of type $14^2 12^1$ with replication number 14. Apply Construction 2.2, filling in the relevent 14- and 12-point designs (Theorem 1.7).

RRP(40, 27).

Twenty parallel classes:

$$\begin{array}{cc} 0_0 5_1 & 0_1 5_0 \\ 1_0 9_0 & 1_1 9_1 \\ 2_0 8_0 14_1 & 2_1 8_1 14_0 \\ 3_0 7_0 4_1 & 3_1 7_1 4_0 \\ 6_0 10_1 19_1 & 6_1 10_0 19_0 \\ 11_0 18_0 & 11_1 18_1 \\ 12_0 17_0 & 12_1 17_1 \\ 13_0 15_0 16_0 & 13_1 15_1 16_1 \end{array} \quad \text{mod } (20, -)$$

Then apply Corollary 2.6.

RRP(40, k), $30 \leqslant k \leqslant 39$. Apply observations (01), (02) and (03) to the above design (with regards to (03) the relevant blocks here are instead $13_0 15_0 16_0$, $2_1 8_1 14_0, \ldots$, which are to be replaced with $13_0 16_0$, $14_0 15_0$, $2_1 8_1, \ldots$).

RRP(44, 22), RRP(44, 25). Take a resolvable TD(3, 15) and remove a point to obtain a resolvable 2, 3-GDD of type $15^2 14^1$ with replication number 15. Apply Construction 2.2, filling in the relevant 15- and 14-point designs (an RRP(15, 10) exists by Theorem 2.7).

RRP(44, 28).

$$\begin{array}{lccl}
 & 0_0 2_1 10_1 & 0_1 2_0 10_0 & \\
 & 1_0 11_0 & 1_1 11_1 & \\
 & 3_0 9_0 & 3_1 9_1 & \\
 & 4_0 8_0 7_1 & 4_1 8_1 7_0 & \\
\text{Twenty-two parallel classes:} & 5_0 12_1 21_1 & 5_1 12_0 21_0 & \bmod (22, -) \\
 & 6_0 14_1 19_1 & 6_1 14_0 19_0 & \\
 & 13_0 20_0 & 13_1 20_1 & \\
 & 15_0 16_0 18_0 & 15_1 16_1 18_1 & \\
 & \multicolumn{2}{c}{17_0 17_1} &
\end{array}$$

Then apply Corollary 2.6.

RRP(44, 31).

$$\begin{array}{lccl}
 & 0_0 2_1 10_1 & 0_1 2_0 10_0 & \\
 & 1_0 11_0 & 1_1 11_1 & \\
 & 3_0 9_0 & 3_1 9_1 & \\
 & 4_0 8_0 7_1 & 4_1 8_1 7_0 & \\
\text{Twenty-two parallel classes:} & 5_0 12_1 21_1 & 5_1 12_0 21_0 & \bmod (22, -) \\
 & 6_0 14_1 19_1 & 6_1 14_0 19_0 & \\
 & 13_0 20_0 & 13_1 20_1 & \\
 & 15_0 18_0 & 15_1 18_1 & \\
 & 16_0 17_0 & 16_1 17_1 &
\end{array}$$

Then apply Corollary 2.6.

RRP(44, k), $34 \leqslant k \leqslant 40$. Apply observations (01) and (02) to the above design.

RRP(46, 24). Remove a block of size two from an NKTS(48).

RRP(46, 27). Take a resolvable TD(3, 18) and remove eight points from one of the groups to obtain a resolvable 2, 3-GDD of type $18^2 10^1$ with replication number 18. Apply Construction 2.2, filling in RRP(18, 9) (e.g. an NKTS(18)) and RRP(10, 9).

RRP(46, 30).

$$\begin{array}{lccl}
 & 1_0 2_1 21_1 & 1_1 2_0 21_0 & \\
 & 22_0 20_1 3_1 & 22_1 20_0 3_0 & \\
 & 4_0 9_1 14_1 & 4_1 9_0 14_0 & \\
 & 19_0 13_1 10_1 & 19_1 13_0 10_0 & \\
\text{Twenty-three parallel classes:} & 5_0 18_0 & 5_1 18_1 & \bmod (23, -) \\
 & 6_0 17_0 & 6_1 17_1 & \\
 & 0_0 7_1 16_1 & 0_1 8_0 15_0 & \\
 & 8_1 15_1 & 7_0 16_0 & \\
 & 11_0 12_0 & 11_1 12_1 &
\end{array}$$

Then apply Corollary 2.6.

RRP(46, k), $33 \leqslant k \leqslant 45$. Apply observations (01) and (02) to the above design.

RRP(50, 25). Remove a point from a KTS(51).

RRP(50, 28), RRP(50, 31). Take a resolvable TD(3, 18) and remove four points from one of the groups to obtain a resolvable 2, 3-GDD of type $18^2 14^1$ with replication number 18. Apply Construction 2.2, filling in the relevant 18- and 14-point designs (Theorem 1.7).

RRP(50, 34).

$$\begin{array}{c} \text{Twenty-five} \\ \text{parallel classes:} \end{array} \quad \begin{array}{cccc} 1_0 2_1 23_1 & 6_0 19_0 & 1_1 2_0 23_0 & 6_1 19_1 \\ 24_0 22_1 3_1 & 0_0 7_1 18_1 & 24_1 22_0 3_0 & 0_1 8_0 17_0 \\ 4_0 9_1 16_1 & 7_0 18_0 & 4_1 9_0 16_0 & 8_1 17_1 \\ 21_0 15_1 10_1 & 11_0 14_0 & 21_1 15_0 10_0 & 11_1 14_1 \\ 5_0 20_0 & 12_0 13_0 & 5_1 20_1 & 12_1 13_1 \end{array} \quad \bmod (25, -)$$

Then apply Corollary 2.6.

RRP(50, k), $37 \leqslant k \leqslant 49$. Apply observations (01) and (02) to the above design.

RRP(52, 27), RRP(52, 30) (and RRP(52, 33)). Take a resolvable TD(3, 18) and remove two points from one of the groups to obtain a resolvable 2, 3-GDD of type $18^2 16^1$ with replication number 18. Apply Construction 2.2, filling in the relevant 18- and 16-point designs (Theorem 1.7).

RRP(52, 33).

$$\begin{array}{c} \text{Twenty-six} \\ \text{parallel classes:} \end{array} \quad \begin{array}{cccc} 0_0 2_1 12_1 & 6_0 14_1 25_1 & 0_1 2_0 12_0 & 6_1 14_0 25_0 \\ 1_0 13_0 & 7_0 16_1 23_1 & 1_1 13_1 & 7_1 16_0 23_0 \\ 3_0 11_0 & 20_0 15_1 24_1 & 3_1 11_1 & 20_1 15_0 24_0 \\ 4_0 10_0 & 17_0 22_0 & 4_1 10_1 & 17_1 22_1 \\ 8_0 5_1 9_1 & 18_0 19_0 21_0 & 8_1 5_0 9_0 & 18_1 19_1 21_1 \end{array} \quad \bmod (26, -)$$

Then apply Corollary 2.6.

RRP(52, k), $36 \leqslant k \leqslant 48$. Apply observations (01), (02) and (03) to the above design.

RRP(56, 28). Remove a point from a KTS(57).

RRP(56, 31), RRP(56, 34). Take a resolvable TD(3, 21) and remove seven points from one of the groups to obtain a resolvable 2, 3-GDD of type $21^2 14^1$ with replication number 21. Apply Construction 2.2, filling in the relevant 21- and 14-point designs (an RRP(21, 13) exists by Theorem 2.7).

RRP(56, 37).

$$\begin{array}{c} \text{Twenty-eight} \\ \text{parallel classes:} \end{array} \quad \begin{array}{cccc} 0_0 2_1 12_1 & 6_0 14_1 27_1 & 0_1 2_0 12_0 & 6_1 14_0 27_0 \\ 1_0 13_0 & 7_0 16_1 25_1 & 1_1 13_1 & 7_1 16_0 25_0 \\ 3_0 11_0 & 21_0 15_1 26_1 & 3_1 11_1 & 21_1 15_0 26_0 \\ 4_0 10_0 & 17_0 24_0 & 4_1 10_1 & 17_1 24_1 \\ 8_0 5_1 9_1 & 18_0 23_0 & 8_1 5_0 9_0 & 18_1 23_1 \\ & 19_0 20_0 22_0 & & 19_1 20_1 22_1 \end{array} \quad \bmod (28, -)$$

Then apply Corollary 2.6.

RRP(56, k), $40 \leqslant k \leqslant 52$. Apply observations (01), (02) and (03) to the above design.

RRP(50, 30), RRP(58, 33) (and RRP(58, 36)). Take a resolvable TD(3, 20) and remove two points from one of the groups to obtain a resolvable 2, 3-GDD of type $20^2 18^1$ with replication number 20. Apply Construction 2.2, filling in the relevant 20- and 18-point designs (Theorem 1.7).

RRP(58, 36).

$$\begin{array}{rccccl}
 & 1_0 2_1 27_1 & 24_0 16_1 13_1 & 1_1 2_0 27_0 & 24_1 16_0 13_0 & \\
 & 28_0 3_1 26_1 & 6_0 23_0 & 28_1 3_0 26_0 & 6_1 23_1 & \\
\text{Twenty-nine} & 4_0 9_1 20_1 & 8_0 21_0 & 4_1 9_0 20_0 & 8_1 21_1 & \text{mod } (29, -) \\
\text{parallel classes:} & 25_0 19_1 10_1 & 11_0 18_0 & 25_1 19_0 10_0 & 11_1 18_1 & \\
 & 5_0 14_1 15_1 & 12_0 17_0 & 5_1 14_0 15_0 & 7_1 22_1 & \\
 & & 0_1 7_0 22_0 & & 0_0 12_1 17_1 &
\end{array}$$

Then apply Corollary 2.6.

RRP(58, k), $39 \leqslant k \leqslant 57$. Apply observations (01) and (02) to the above design.

RRP(62, 31), RRP(62, 34), RRP(62, 37). Take a resolvable TD(3, 21) and remove a point to obtain a resolvable 2, 3-GDD of type $21^2 20^1$ with replication number 21. Apply construction 2.2, filling in the relevant 21- and 20-point designs (see Theorem 2.7 and Lemma 2.8).

RRP(62, 40).

$$\begin{array}{rccccl}
 & 1_0 2_1 29_1 & 0_0 7_1 24_1 & 1_1 2_0 29_0 & 0_1 8_0 23_0 & \\
 & 30_0 3_1 28_1 & 6_0 25_0 & 30_1 3_0 28_0 & 6_1 25_1 & \\
\text{Thirty-one} & 4_0 9_1 22_1 & 11_0 20_0 & 4_1 9_0 22_0 & 11_1 21_1 & \text{mod } (31, -) \\
\text{parallel classes:} & 27_0 21_1 10_1 & 12_0 19_0 & 27_1 21_0 10_0 & 12_1 19_1 & \\
 & 5_0 14_1 17_1 & 13_0 18_0 & 5_1 14_0 17_0 & 13_1 18_1 & \\
 & 26_0 15_1 16_1 & 7_0 24_0 & 26_1 15_0 16_0 & 8_1 23_1 &
\end{array}$$

Then apply Corollary 2.6.

RRP(62, k), $43 \leqslant k \leqslant 61$. Apply observations (01) and (02) to the above design.

RRP(64, 33). Remove a block of size two from an NKTS (66).

RRP(64, 36), RRP(64, 39). Take a resolvable TD(3, 24) and remove eight points from one of the groups to obtain a resolvable 2, 3-GDD of type $24^2 16^1$ with replication number 24. Apply Construction 2.2, filling in the relevant 24- and 16-point designs (Theorem 1.7).

RRP(64, 42).

Thirty-two parallel classes:

$$\begin{array}{cccccc}
0_2 2_1 14_1 & 7_0 16_1 31_1 & & 0_1 2_0 14_0 & 7_1 16_0 31_0 & \\
20_0 1_1 15_1 & 8_0 18_1 29_1 & & 20_1 1_0 15_0 & 8_1 18_0 29_0 & \\
3_0 13_0 & 24_0 17_1 30_1 & & 3_1 13_1 & 24_1 17_0 30_0 & \\
4_0 12_0 & 19_0 28_0 & & 4_1 12_1 & 19_1 28_1 & \mod (32, -) \\
5_0 11_0 & 21_0 26_0 & & 5_1 11_1 & 21_1 26_1 & \\
9_0 6_1 10_1 & 22_0 23_0 25_0 & & 9_1 6_0 10_0 & 22_1 23_1 25_1 & \\
 & & 27_0 27_1 & & &
\end{array}$$

Then apply Corollary 2.6.

RRP(64, 45).

Thirty-two parallel classes:

$$\begin{array}{ccccc}
0_0 2_1 14_1 & 7_0 16_1 31_1 & 0_1 2_0 14_0 & 7_1 16_0 31_0 & \\
1_1 15_1 & 8_0 18_1 29_1 & 1_0 15_0 & 8_1 18_0 29_0 & \\
3_0 13_0 & 24_0 17_1 30_1 & 3_1 13_1 & 24_1 17_0 30_0 & \\
4_0 12_0 & 19_0 28_0 & 4_1 12_1 & 19_1 28_1 & \mod (32, -) \\
5_0 11_0 & 21_0 26_0 & 5_1 11_1 & 21_1 26_1 & \\
9_0 6_1 10_1 & 22_0 23_0 25_0 & 9_1 6_0 10_0 & 22_1 23_1 25_1 & \\
 & 20_0 27_0 & & 20_1 27_1 &
\end{array}$$

Then apply Corollary 2.6.

RRP(64, k), $48 \leqslant k \leqslant 60$. Apply observations (01), (02) and (03) to the above design.

RRP(68, 34). Remove a point from a KTS(69).

RRP(68, 37), RRP(68, 40), RRP(68, 43). Take a resolvable TD(3, 24) and remove four points from one of the groups to obtain a resolvable 2, 3-GDD of type $24^2 20^1$ with replication number 24. Apply Construction 2.2, filling in the relevant 24- and 20-point designs (Theorem 1.7).

RRP(68, 43).

Thirty-four parallel classes:

$$\begin{array}{ccccc}
0_0 2_1 16_1 & 7_0 11_0 & 0_1 2_0 16_0 & 7_1 11_1 & \\
21_0 1_1 17_1 & 8_0 18_1 33_1 & 21_1 1_0 17_0 & 8_1 18_0 33_0 & \\
3_0 15_0 & 9_0 20_1 31_1 & 3_1 15_1 & 9_1 20_0 31_0 & \\
4_0 14_0 & 26_0 19_1 32_1 & 4_1 14_1 & 26_1 19_0 32_0 & \mod (34, -) \\
10_0 5_1 13_1 & 30_0 22_1 29_1 & 10_1 5_0 13_0 & 30_1 22_0 29_0 & \\
6_0 12_0 & 23_0 28_0 & 6_1 12_1 & 23_1 28_1 & \\
 & 24_0 25_0 27_0 & & 24_1 25_1 27_1 &
\end{array}$$

Then apply Corollary 2.6.

RRP(68, k), $46 \leqslant k \leqslant 64$. Apply observations (01), (02) and (03) to the above design.

RRP(70, 36), RRP(70, 39), RRP(70, 42), RRP(70, 45). Take a resolvable TD(3, 24) and remove two points from one of the groups to obtain a resolvable

2, 3-GDD of type $24^2 22^1$ with replication number 24. Apply Construction 2.2, filling in the relevant 24- and 22-point designs (Theorem 1.7).

RRP(70, 48).

Thirty-five parallel classes:	$0_0 7_1 28_1$	$30_0 15_1 20_1$	$0_1 8_0 27_0$	$30_1 15_0 20_0$	mod (35, –)
	$7_0 28_0$	$6_0 29_0$	$8_1 27_1$	$6_1 29_1$	
	$1_0 2_1 33_1$	$11_0 24_0$	$1_1 2_0 33_0$	$11_1 24_1$	
	$34_0 3_1 32_1$	$12_0 23_0$	$34_1 3_0 32_0$	$12_1 23_1$	
	$4_0 9_1 26_1$	$13_0 22_0$	$4_1 9_0 26_0$	$13_1 22_1$	
	$31_0 10_1 25_1$	$16_0 19_0$	$31_1 10_0 25_0$	$16_1 19_1$	
	$5_0 14_1 21_1$	$17_0 18_0$	$5_1 14_0 21_0$	$17_1 18_1$	

Then apply Corollary 2.6.

RRP(70, k), $51 \leq k \leq 69$. Apply observations (01) and (02) to the above design.

RRP(80, 40), RRP(80, 43), RRP(80, 46), RRP(80, 49). Take a resolvable TD(3, 27) and remove a point to obtain a resolvable 2, 3-GDD of type $27^2 26^1$ with replication number 27. Apply Construction 2.2, filling in the relevant 27- and 26-point designs (see Theorem 2.7 and Lemma 2.8).

RRP(80, 52). Apply Theorem 2.3 to a 4-GDD of type 3^4 (i.e. a TD(4, 3)) to obtain a 2, 3-frame of type 18^4 in which all holes have degree 12. Now use Construction 2.4 (with $w = 8$ and $d = 4$), filling the holes with the RRP(26, 16) constructed in the appendix (noting that this design has a sub-RRP(8, 4)).

RRP(80, 55).

Forty parallel classes:	$0_0 2_1 18_1$	$28_0 29_0 31_0$	$0_1 2_0 18_0$	$28_1 29_1 31_1$	mod (40, –)
	$10_0 22_1 37_1$	$30_0 21_1 38_1$	$10_1 22_0 37_0$	$30_1 21_0 38_0$	
	$9_0 20_1 39_1$	$24_0 17_1 3_1$	$9_1 20_0 39_0$	$24_1 17_0 3_0$	
	$11_0 12_1 8_1$	$35_0 19_1 1_1$	$11_1 12_0 8_0$	$35_1 19_0 1_0$	
	$4_0 16_0$	$23_0 36_0$	$4_1 16_1$	$23_1 36_1$	
	$5_0 15_0$	$25_0 34_0$	$5_1 15_1$	$25_1 34_1$	
	$6_0 14_0$	$26_0 33_0$	$6_1 14_1$	$26_1 33_1$	
	$7_0 13_0$	$27_0 32_0$	$7_1 13_1$	$27_1 32_1$	

Then apply Corollary 2.6.

RRP(80, k), $58 \leq k \leq 76$. Apply observations (01), (02) and (03) to the above design.

RRP(82, 42). Remove a block of size two from an NKTS(84).

RRP(82, 45), RRP(82, 48), RRP(82, 51). Take a resolvable TD(3, 30) and remove eight points from one of its groups to obtain a resolvable 2, 3-GDD of type $30^2 22^1$ with replication number 30. Apply Construction 2.2, filling in the relevant 30- and 22-point designs (Theorem 1.7).

RRP(82, 54).

	$1_0 2_1 39_1$	$5_0 20_1 21_1$	$1_1 2_0 39_0$	$5_1 20_0 21_0$	
	$40_0 38_1 3_1$	$36_0 28_1 13_1$	$40_1 38_0 3_0$	$36_1 28_0 13_0$	
	$4_0 9_1 32_1$	$6_0 15_1 26_1$	$4_1 9_0 32_0$	$6_1 15_0 26_0$	
Forty-one	$37_0 31_1 10_1$	$35_0 25_1 16_1$	$37_1 31_0 10_0$	$35_1 25_0 16_0$	mod (41, –)
parallel classes:	$0_0 11_1 30_1$	$14_0 27_0$	$0_1 12_0 29_0$	$14_1 27_1$	
	$11_0 30_0$	$17_0 24_0$	$12_1 29_1$	$17_1 24_1$	
	$7_0 34_0$	$18_0 23_0$	$7_1 34_1$	$18_1 23_1$	
	$8_0 33_0$	$19_0 22_0$	$8_1 33_1$	$19_1 22_1$	

Then apply Corollary 2.6.

RRP(82, *k*), 57 ≤ *k* ≤ 81. Apply observations (01) and (02) to the above design.

RRP(86, 43). Remove a point from a KTS(87).

RRP(86, 46), RRP(86, 49), RRP(86, 52), RRP(86, 55). Take a resolvable TD(3, 30) and remove four points from one of the groups to obtain a resolvable 2, 3-GDD of type $30^2 26^1$. Apply Construction 2.2, filling in the relevant 30- and 26-point designs (Theorem 1.7).

RRP(86, 58).

	$1_0 2_1 41_1$	$5_0 12_1 31_1$	$1_1 2_0 41_0$	$5_1 12_0 31_0$	
	$42_0 40_1 3_1$	$38_0 30_1 13_1$	$42_1 40_0 3_0$	$38_1 30_0 13_0$	
	$4_0 9_1 34_1$	$6_0 15_1 28_1$	$4_1 9_0 34_0$	$6_1 15_0 28_0$	
Forty-three	$39_0 33_1 10_1$	$37_0 17_1 26_1$	$39_1 33_0 10_0$	$37_1 17_0 26_0$	
parallel classes:	$0_0 16_1 27_1$	$11_0 32_0$	$0_1 19_0 24_0$	$11_1 32_1$	mod (43, –)
	$16_0 27_0$	$14_0 29_0$	$19_1 24_1$	$14_1 29_1$	
	$7_0 36_0$	$18_0 25_0$	$7_1 36_1$	$18_1 25_1$	
	$8_0 35_0$	$20_0 23_0$	$8_1 35_1$	$20_1 23_1$	
		$21_0 22_0$		$21_1 22_1$	

Then apply Corollary 2.6.

RRP(86, *k*), 61 ≤ *k* ≤ 85. Apply observations (01) and (02) to the above design.

RRP(88, 45), RRP(88, 48), RRP(88, 51), RRP(88, 54), RRP(88, 57). Take a resolvable TD(3, 30) and remove two points from one of its groups to obtain a resolvable 2, 3-GDD of type $30^2 28^1$ with replication number 30. Apply Construction 2.2, filling in the relevant 30- and 28-point designs (Theorem 1.7).

RRP(88, 60).

	$0_0 2_1 20_1$	$1_0 15_1 17_1$	$0_1 2_0 20_0$	$1_1 5_0 17_0$	
	$10_0 22_1 43_1$	$21_0 6_1 16_1$	$10_1 22_0 43_0$	$21_1 6_0 16_0$	
	$11_0 24_1 41_1$	$31_0 32_0 34_0$	$11_1 24_0 41_0$	$31_1 32_1 34_1$	
	$12_0 13_1 9_1$	$33_0 23_1 42_1$	$12_1 13_0 9_0$	$33_1 23_0 42_0$	
Forty-four	$4_0 18_0$	$26_0 39_0$	$4_1 18_1$	$26_1 39_1$	mod (44, –)
parallel classes:	$7_0 15_0$	$27_0 38_0$	$7_1 15_1$	$27_1 38_1$	
	$8_0 14_0$	$28_0 37_0$	$8_1 14_1$	$28_1 37_1$	
	$19_0 25_1 40_1$	$29_0 36_0$	$19_1 25_0 40_0$	$29_1 36_1$	
		$30_0 35_0$		$30_1 35_1$	

$$3_0 3_1$$

Then apply Corollary 2.6.

RRP(88, 63).

	$0_0 2_1 20_1$	$1_0 5_1 17_1$	$0_1 2_0 20_0$	$1_1 5_0 17_0$	
	$10_0 22_1 43_1$	$21_0 6_1 16_1$	$10_1 22_0 43_0$	$21_1 6_0 16_0$	
	$11_0 24_1 41_1$	$31_0 32_0 34_0$	$11_1 24_0 41_0$	$31_1 32_1 34_1$	
Forty-four	$12_0 13_1 9_1$	$33_0 23_1 42_1$	$12_1 13_0 9_0$	$33_1 23_0 42_0$	
parallel classes:	$4_0 18_0$	$26_0 39_0$	$4_1 18_1$	$26_1 39_1$	mod (44, –)
	$7_0 15_0$	$27_0 38_0$	$7_1 15_1$	$27_1 38_1$	
	$8_0 14_0$	$28_0 37_0$	$8_1 14_1$	$28_1 37_1$	
	$3_0 19_0$	$29_0 36_0$	$3_1 19_1$	$29_1 36_1$	
	$25_1 40_1$	$30_0 35_0$	$25_0 40_0$	$30_1 35_1$	

Then apply Corollary 2.6.

RRP(88, k), $66 \leq k \leq 84$. Apply observations (01), (02) and (03) to the above design.

4. Extension to R*RP's

In this section we will apply frames to prove the following result.

Theorem 4.1. *Let p be an even integer, $p \geq 96$. There exists an* R*RP(p, k) *for every k with $p/2 \leq k \leq p-1$, except where $p \equiv 4$ modulo 6 and $k = p/2$.*

In our designs the "aberrant" blocks will have size 1 (viewed as tournament schedules, this corresponds to assigning byes in at most two rounds of the tournament) and where there are two such blocks they will not cover the same point (i.e. no single player is assigned two byes). For the remainder of this section the notation R*RP(p, k) (and R*RP(p, k) – RRP(w, d)) will be understood to mean a design with these additional properties.

Proof of Theorem 4.1. From Theorem 1.7 we may asume that $p \equiv 2$ or 4 modulo 6. Suppose first that $p \equiv 2$ modulo 6. Since $p \geqslant 98$ we can write $p = 6s + 2$ where $s \geqslant 16$; from (the proof of) Theorem 3.3 there is a 4, 5-GDD of type $4^t r^1$ on s points, from some $t > 0$ and $r \geqslant 0$.

We may assume that $k \not\equiv 1$ modulo 3, else $p(k - p + 1) \equiv 0$ modulo 3 and we already know that an RRP(p, k) exists (Section 3). Thus we have two cases to consider

(i) $k \equiv 2$ modulo 3. Let $q = \frac{1}{3}(k - ((p/2) + 1))$ and let $n_1, \ldots, n_t, n_\infty$ be a sequence of integers with $n_1 \in \{0, 3\}$, $0 \leqslant n_i \leqslant 4$ for $i = 2, \ldots, t$ and $0 \leqslant n_\infty \leqslant r$ satisfying $n_\infty + \sum n_i = q$. Now apply Theorem 2.3 to construct a 2, 3-frame of type $24^t(6r)^1$ in which the ith hole H_i of size 24 has degree $12 + 3n_i + 1$ if $i = 1$, or $12 + 3n_i$ if $2 \leqslant i \leqslant t$, and in which the hole H_∞ of size $6r$ has degree $3r + 3n_\infty$. Add two 'ideal' points to this frame, and apply Construction 2.4 with $w = 2$ and $d = 1$. On the points of H_1 (together with the ideal points) construct an R*RP$(26, 12 + 3n_1 + 2)$ – RRP(2, 1) and on the remaining H_i construct RRP$(26, 12 + 3n_i + 1)$ – RRP(2, 1) (see appendix). On H_∞ construct an RRP$(6r + 2, 3r + 3n_\infty + 1)$ (Section 3). We obtain an R*RP on p points with replication number $3r + 1 + 3n_\infty + 1 + \sum (12 + 3n_i) = 2 + (p - 2)/2 + 3q = k$ as desired.

(ii) $k \equiv 0$ modulo 3. Let $q = \frac{1}{3}(k - ((p/2) + 2))$ and proceed as above, except that the hole H_1 is to have degree $12 + 3n_1 + 2$ in the frame, and on this hole we construct an R*RP$(26, 12 + 3n_1 + 3)$ – RRP(2, 1) (see appendix).

The proof for the case $p \equiv 4$ modulo 6 is similar. Here we write $p = 6s + 4$ where again $s \geqslant 16$. The relevant cases to consder are:

(i)′ $k = 1$ modulo 3. Let $q = \frac{1}{3}(k - ((p/2) + 2))$ and construct the frame as in Case (i). Add four 'ideal' points and apply Construction 2.4 with $w = 4$ and $d = 3$, filling in R*RP$(28, 12 + 3n_1 + 4)$ – RRP(4, 3) and RRP$(28, 12 + 3n_i + 3)$ – RRP(4, 3) (appendix) and an RRP$(6r + 4, 3r + 3n_\infty + 3)$ (Section 3).

(ii)′ $k \equiv 2$ modulo 3. We have already noted in the introduction that k cannot be $p/2$ (see Lemma 1.9). Thus let $q = \frac{1}{3}(k - ((p/2) + 3))$ and construct the frame as in Case (ii) (i.e. giving hole H_1 degree $12 + 3n_1 + 2$) and then proceed as in Case (i)′, except that hole H_1 is to be filled in with an R*RP$(28, 12 + 3n_1 + 5)$ – RRP(4, 3) (appendix).

This completes the proof of Theorem 4.1. □

5. Conclusion

Theorems 1.7, 1.8 and 4.1 imply that where $2k + 1 \leqslant v \leqslant 3k$ the Stinson bound (Theorems 1.2 and 1.5) is sharp for $g^{(k)}(v)$ whenever $v - k$ is even ($v \neq 3k$ when $k \equiv 2$ modulo 3), and either

(i) $(v - k)(2k - v + 1) \equiv 0$ modulo 3 ($(v, k) \neq (9, 3)$ or $(18, 6)$), or

(ii) $v - k \geqslant 96$.

Moreover, there exist optimal configurations with block sizes from $\{2, 3, 4, k\}$.

In the second of this series (The Existence of Restricted Resolvable Designs II: $(1,2)$-Factorizations of K_{2n+1}) we will prove that the necessary conditions for the existence of odd-ordered RRP(p, k)'s are sufficient, except possibly for eighteen values of p.

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APPENDIX

20-point RRP's 'missing' a sub-RRP(2, 1)
These are just 'ordinary' RRP's.

RRP(20, 10). Remove a point from a KTS(21).

RRP(20, 13).

$$\begin{array}{ll} 2_04_03_1 & 0_03_0 \\ 7_08_00_1 & 1_05_0 \\ 6_02_14_1 & 1_15_1 \\ 9_06_19_1 & 7_18_1 \end{array} ; \quad \begin{array}{l} 0_05_0 \\ 0_15_1 \end{array} ; \quad 0_04_1; \quad 0_05_1 \qquad \bmod (10, -)$$

RRP(20, 16).

$$\begin{array}{llllll} 0_00_1 & 7_08_0 & 0_05_0 & & & \\ 1_05_03_1 & 1_15_1 & 0_15_1; & 0_01_1; & 0_04_1; & \\ 3_09_16_1 & 7_18_1; & & & & \bmod (10, -) \\ 6_09_0 & 2_14_1 & & & & \\ 2_04_0 & & & & & \\ & & 0_05_1; & 0_07_1; & 0_09_1 & \end{array}$$

RRP(20, 19). This is a 1-factorization of K_{20}.

26-point RRP's "missing" a sub-RRP(2, 1).
These are just 'ordinary' RRP's.

To construct an RRP(26, k), apply Theorem 2.3 to a 4-GDD of type 1^4 to obtain a 2, 3-frame of type 6^4 in which there are $\frac{1}{3}(k-13)$ holes of degree 6 and $4-\frac{1}{3}(k-13)$ holes of degree 3. Now use Construction 2.4 (with $w=2$ and $d=1$) using RRP(8, 4) and RRP(8, 7) as input designs.

22-point RRP's 'missing' a sub-RRP(4, 3).

RRP(22, 12)-RRP(4, 3). Our point set is $\{a, b, c, d\} \cup (\mathbb{Z}_6 \times \mathbb{Z}_3)$.

$$\text{Six parallel classes:} \quad \begin{array}{ll} 0_11_12_2 & 2_05_2b \\ 3_00_14_2 & 4_03_2 \\ 1_05_1a & 2_11_2c \\ 5_04_1d & 3_10_2 \end{array} \quad \bmod (6, -)$$

The remaining three parallel classes are

$$\begin{array}{ll} 0_02_14_2 & 3_05_11_2 \\ 1_02_0 & 4_05_0c \\ 3_14_1 & 0_11_1b \\ 5_20_2d & 2_23_2a \end{array} ; \quad \begin{array}{ll} 2_04_10_2 & 5_01_13_2 \\ 3_04_0 & 0_01_0c \\ 5_10_1 & 2_13_1b \\ 1_22_2d & 4_25_2a \end{array} ; \quad \begin{array}{ll} 4_00_12_2 & 1_03_15_2 \\ 5_00_0 & 2_03_0c \\ 1_12_1 & 4_15_1b \\ 3_24_2d & 0_21_2a \end{array}$$

Three 'holey' parallel classes:

$$\begin{array}{llll} & 0_02_04_0 & 0_03_0 & \\ 0_00_10_2; & 0_12_14_1; & 0_13_1 & \mathrm{mod}\,(6,-) \\ & 0_22_24_2 & 0_23_2 & \end{array}$$

Remark: adding two extra points e, f and changing the blocks 4_03_2, 3_10_2, 1_02_0, 3_04_0, 5_00_0, 3_14_1, 5_10_1, 1_12_1 to 4_03_2e, 3_10_2f, 1_02_0f, 3_04_0f, 5_00_0f, 3_14_1e, 5_10_1e, 1_12_1e yields an NKTS(24) − NKTS(6).

RRP(22, 15)-RRP(4, 3). Our point set is $\{a, b, c, d\} \cup (\mathbb{Z}_6 \times \mathbb{Z}_3)$.
Twelve parallel classes are given by

$$\begin{array}{lllll} a0_03_1 & 2_05_2 & a4_2 & 0_01_12_2 & \\ b1_02_2 & 4_13_2 & b2_1 & 1_03_15_2 & \\ c0_14_2 & 3_05_0; & c5_0 & 2_01_2 & \mathrm{mod}\,(6,-) \\ d4_02_1 & 1_15_1 & d0_2 & 0_13_2 & \\ & 0_21_2 & 4_15_1 & 3_04_0 & \end{array}$$

and three 'holey' parallel classes by

$$\begin{array}{llll} & 0_05_1 & 0_03_0 & \\ 0_00_10_2; & 0_22_24_2; & 0_13_1 & \mathrm{mod}\,(6,-) \\ & & 0_23_2 & \end{array}$$

RRP(22, 18)-RRP(4, 3). Our point set is $\{a, b, c, d\} \cup (\mathbb{Z}_6 \times \mathbb{Z}_3)$.
Twelve parallel classes:

$$\begin{array}{lllllll} 0_01_1 & 2_13_2 & b5_0 & 0_03_1 & 5_04_1 & b3_2 & \\ 1_03_1 & 5_11_2 & c4_0; & 1_02_2 & 0_14_2 & c5_1 & \mathrm{mod}\,(6,-) \\ 2_00_2 & 2_24_2 & d0_1 & 2_05_2 & 2_11_2 & d4_0 & \\ 3_05_2 & a4_1 & & 3_01_1 & a0_2 & & \end{array}$$

The remaining three parallel classes are

$$\begin{array}{llllll} 0_00_10_2 & a4_05_0 & 2_02_12_2 & a0_01_0 & 4_04_14_2 & a2_03_0 \\ 3_03_13_2 & b4_15_1. & 5_05_15_2 & b0_11_1. & 1_01_11_2 & b2_13_1 \\ 1_02_0 & c4_25_2, & 3_04_0 & c0_21_2, & 5_00_0 & c2_23_2 \\ 1_12_1 & d1_22_2 & 3_14_1 & d3_24_2 & 5_10_1 & d5_20_2 \end{array}$$

and the three 'holey' parallel classes are

$$\begin{array}{llll} 0_02_04_0. & 0_12_14_1. & 0_03_0 & \\ 0_13_2\ , & 0_05_2\ , & 0_13_1 & \mathrm{mod}\,(6,-) \\ & & 0_23_2 & \end{array}$$

RRP(22, 21)-RRP(4, 3). Our point set is $\{a, b, c, d\} \cup \mathbb{Z}_{18}$

Eighteen parallel classes:	$a, 0$	$1, 9$	$10, 17$	mod 18
	$b, 5$	$2, 8$	$11, 16$	
	$c, 13$	$3, 7$	$12, 15$	
	$d, 14$	$4, 6$		

Three 'holey' parallel classes are obtained by developing the blocks $0, 9$ and $0, 1 \mod 18$.

28-point RRP's 'missing' a sub-RRP(4, 3).

RRP(28, 15)-RRP(4, 3). Our point set is $\{a, b, c, d\} \cup (\mathbb{Z}_{12} \times \mathbb{Z}_2)$

Twelve parallel classes:	$a4_09_1$	11_011_1	mod (12, –)
	$b6_03_1$	$0_01_03_0$	
	$c7_01_1$	$0_12_15_1$	
	$d8_07_1$	$5_010_08_1$	
	9_04_1	$2_06_110_1$	

Two classes of triples on $\mathbb{Z}_{12} \times \mathbb{Z}_2$ are obtained by developing $0_01_12_1$ and $1_05_09_0$ mod(12, –); a class of pairs is obtained by developing 0_06_0 and 0_16_1 mod(12, –). (*Remark*: adding two extra points e and f and changing the base blocks 9_04_1 and 11_011_1 to $e9_04_1$ and $f11_011_1$ yields an NKTS(30) – NKTS(6)).

RRP(28, 18)-RRP(4, 3). Our point set is $\{a, b, c, d\} \cup (\mathbb{Z}_8 \times \mathbb{Z}_3)$.
Nine parallel classes:

$a3_0$	0_11_1	$4_04_14_2$	$a5_0$	4_15_1	$0_00_10_2$	$a3_1$	0_02_0	$3_04_15_2$
$b2_0$	2_13_1	$5_05_15_2.$	$b4_0$	6_17_1	$1_01_11_2.$	$b2_1$	1_07_0	$4_05_16_2.$
$c1_0$	0_21_2	$6_06_16_2$'	$c7_0$	4_25_2	$2_02_12_2$'	$c1_1$	1_23_2	$5_06_17_2$'
$d0_0$	2_23_2	$7_07_17_2$	$d6_0$	6_27_2	$3_03_13_2$	$d0_1$	2_24_2	$6_07_10_2$

$a5_1$	4_06_0	$7_00_11_2$	$a3_2$	4_05_0	$0_02_14_2$	$a5_2$	0_01_0	$4_06_10_2$
$b4_1$	5_03_0	$0_01_12_2.$	$b2_2$	6_07_0	$1_03_15_2.$	$b4_2$	2_03_0	$5_07_11_2.$
$c7_1$	5_27_2	$1_02_13_2$'	$c1_2$	0_16_1	$2_04_16_2$'	$c7_2$	4_12_1	$6_00_12_2$'
$d6_1$	6_20_2	$2_03_14_2$	$d0_2$	1_17_1	$3_05_17_2$	$d6_2$	5_13_1	$7_01_13_2$

$a0_04_0$	0_12_1	$a0_14_1$	0_06_0	$a0_24_2$	0_03_0
$b1_05_0$	1_13_1	$b1_15_1$	1_03_0	$b1_25_2$	1_02_0
$c2_06_0$	4_16_1	$c2_16_1$	2_04_0	$c2_26_2$	4_07_0
$d3_07_0$	$5_17_1.$	$d3_17_1$	$5_07_0.$	$d3_27_2$	5_06_0
	0_22_2'		0_23_2'		0_13_1
	1_27_2		2_21_2		1_12_1
	4_26_2		4_27_2		4_17_1
	3_25_2		5_26_2		5_16_1

Six more parallel classes are obtained by letting i run through $\mathbb{Z}_8$ in each of the following.

$$\begin{array}{cccccccccccc}
a1_06_0 & & a1_16_1 & & a1_26_2 & & a2_07_0 & & a2_17_1 & & a2_27_2 \\
b0_07_0 & & b0_17_1 & & b0_27_2 & & b3_06_0 & & b3_16_1 & & b3_26_2 \\
c3_04_0 & ; & c3_14_1 & ; & c3_24_2 & ; & c0_05_0 & ; & c0_15_1 & ; & c0_25_2 \\
d2_05_0 & & d2_15_1 & & d2_25_2 & & d1_04_0 & & d1_14_1 & & d1_24_2 \\
i_1(i+3)_2 & & i_0(i+6)_2 & & i_0(i+3)_1 & & i_1(i+4)_2 & & i_0(i+7)_2 & & i_0(i+7)_1
\end{array}$$

The three "holey" parallel classes are

$$0_04_11_2;\quad 0_05_13_2;\quad 0_06_15_2 \quad \mathrm{mod}\,(8,-).$$

RRP(28, 21)-RRP(4, 3). Our point set is $\{a, b, c, d\} \cup (\mathbb{Z}_8 \times \mathbb{Z}_3)$.
The first six parallel classes are as in the previous design, except that in the first, third and fifth classes change $a3_j$, $b2_j$, $c1_j$, $d0_j$ to $a2_j$, $b3_j$, $c0_j$, $d1_j$. The remaining twelve parallel classes are obtained by letting i run through $\mathbb{Z}_8$ in each of the following.

$$\begin{array}{cccccccccccc}
a3_0 & & a3_1 & & a3_2 & & a7_0 & & a7_1 & & a7_2 & \\
b6_0 & & b6_1 & & b6_2 & & b2_0 & & b2_1 & & b2_2 & \\
c1_0 & & c1_1 & & c1_2 & & c5_0 & & c5_1 & & c5_2 & \\
d4_0 & ; & d4_1 & ; & d4_2 & ; & d0_0 & ; & d0_1 & ; & d0_2 & ; \\
0_05_0 & & 0_15_1 & & 0_25_2 & & 4_01_0 & & 4_11_1 & & 4_21_2 & \\
2_07_0 & & 2_17_1 & & 2_27_2 & & 6_03_0 & & 6_13_1 & & 6_23_2 & \\
i_1(i+3)_2 & & i_0(i+6)_2 & & i_0(i+3)_1 & & i_1(i+5)_2 & & i_0(i+1)_2 & & i_0(i+4)_1 &
\end{array}$$

$$\begin{array}{ccccccccccc}
a0_04_0 & & a0_14_1 & & a0_24_2 & & a1_06_0 & & a1_16_1 & & a1_26_2 \\
b1_05_0 & & b1_15_1 & & b1_25_2 & & b0_07_0 & & b0_17_1 & & b0_27_2 \\
c2_06_0 & ; & c2_16_1 & ; & c2_26_2 & ; & c3_04_0 & ; & c3_14_1 & ; & c3_24_2 \\
d3_07_0 & & d3_17_1 & & d3_27_2 & & d2_05_0 & & d2_15_1 & & d2_25_2 \\
i_1(i+6)_2 & & i_0(i+3)_2 & & i_0(i+5)_1 & & i_1(i+4)_2 & & i_0(i+7)_2 & & i_0(i+7)_1
\end{array}$$

The three "holey" parallel classes are

$$\begin{array}{cccccccccc}
0_06_0 & 0_12_1 & 0_22_2 & & 6_05_0 & 2_11_1 & 2_21_2 & & 4_02_11_2 & 0_06_15_2 \\
5_07_0 & 1_13_1 & 1_27_2 & ; & 7_04_0 & 3_10_1 & 7_24_2 & ; & 1_07_16_2 & 5_03_12_2 \\
4_02_0 & 4_16_1 & 4_26_2 & & 2_01_0 & 6_15_1 & 6_25_2 & & 2_00_17_2 & 6_04_13_2 \\
1_03_0 & 5_17_1 & 5_23_2 & & 3_00_0 & 7_14_1 & 3_20_2 & & 3_01_10_2 & 7_05_14_2
\end{array}$$

RRP(28, 24)-RRP(4, 3). Our point set is $\{a_1, a_2, \ldots, a_{10}\} \cup \mathbb{Z}_{18}$

$$\text{Eighteen parallel classes:}\quad \begin{array}{ll}
a_10 & a_814 \\
a_21 & a_916 \\
a_35 & a_{10}17 \\
a_49 & 2,\ 8 \\
a_510 & 3,\ 7 \\
a_611 & 4,\ 6 \\
a_713 & 12,\ 15
\end{array} \quad \mathrm{mod}\ 18$$

Six classes on $\mathbb{Z}_{18}$ are obtained by developing $0, 1, 8$ (for three classes), $0, 5$ and $0, 9 \mod 18$. Now construct an RRP(10, 6) on $a_1, \ldots, a_{10}$ so that $\{a_1, a_4\}$ is a block of size 2. Then the resulting RRP(28, 24) has a sub-RRP(4, 3) on the points $a_1, a_4, 0, 9$. Now just delete the blocks in this subdesign.

RRP(28, 27)-RRP(4, 3). Our point set is $\{a, b, c, d\} \cup \mathbb{Z}_{24}$.

$$\text{Twenty-four parallel classes:} \quad \begin{array}{cccc} a, 0 & 1, 11 & 5, 7 & 15, 20 \\ b, 6 & 2, 10 & 12, 23 & 16, 19 \\ c, 17 & 3, 9 & 13, 22 & \\ d, 18 & 4, 8 & 14, 21 & \end{array} \quad \mod 24$$

Three 'holey' parallel classes are obtained by developing the blocks $0, 12$ and $0, 1 \mod 24$.

26-point R*RP's 'missing' a sub-RRP(2, 1).
These are just 'ordinary' R*RP's.

R*RP(26, 14). From Theorem 2.7 there is an RRP(27, 14). In this design each point is contained in two blocks of size 2; just remove a point.

R*RP(26, 15). Our point set is $\{a_1, \ldots, a_6\} \cup (\mathbb{Z}_{10} \times \mathbb{Z}_2)$.

$$\text{Ten parallel classes:} \quad \begin{array}{ll} a_1 0_0 0_1 & a_6 9_0 7_1 \\ a_2 1_0 2_1 & 2_0 1_1 4_1 \\ a_3 3_0 6_1 & 4_0 8_1 9_1 \\ a_4 7_0 3_1 & 5_0 6_0 \\ a_5 8_0 5_1 & \end{array} \quad \mod (10, -)$$

The remaining five classes are:

$$\begin{array}{lllcll} 0_0 5_0 & 0_1 5_1 & a_1 a_2 & & 0_0 3_0 7_0 & 6_1 8_1 \\ 7_0 9_0 & 2_1 8_1 & a_3 a_4 & & 1_0 4_0 6_0 & 5_1 9_1 \\ 1_0 3_0 & 4_1 6_1 & a_5 a_6; & & 5_0 9_0 & 1_1 3_1 \\ 4_0 8_0 & 3_1 7_1 & & & 2_0 8_0 & a_5 a_4; \\ 2_0 6_0 & 1_1 9_1 & & & 2_1 7_1 & a_3 a_6 \\ & & & & 0_1 4_1 & a_5 a_2 \end{array}$$

$$\begin{array}{lllllll} 3_0 6_0 9_0 & 0_1 8_1 & 0_0 6_0 8_0 & 0_1 2_1 & 2_0 5_0 7_0 & 8_1 3_1 & a_5 a_4 \\ 1_0 5_0 8_0 & 7_1 1_1 & 2_0 4_0 9_0 & 3_1 9_1 & 0_0 4_0 & 0_1 6_1 & a_1 a_3 \\ 0_0 2_0 & 3_1 5_1 & 3_0 5_0 & 5_1 7_1 & 1_0 9_0 & 2_1 4_1 & a_2 a_6 \\ 4_0 7_0 & a_1 a_6; & 1_0 7_0 & a_3 a_2; & 3_0 8_0 & 1_1 5_1 & \\ 4_1 9_1 & a_3 a_5 & 6_1 1_1 & a_1 a_5 & 6_0 & 7_1 9_1 & \\ 2_1 6_1 & a_2 a_4 & 4_1 8_1 & a_4 a_6 & & & \end{array}$$

R*RP(26, 23), R*RP(26, 24). These designs are constructed in [15].

28-point R*RP's "missing" a sub-RRP(4, 3).

R*RP(28, 16)-RRP(4, 3). Our point set is $\{a_1, \ldots, a_7\} \cup (\mathbb{Z}_3 \times \mathbb{Z}_7)$.
Twelve parallel classes are obtained by developing each of the following four classes modulo (3, –).

$a_1 2_3 2_4$	$a_6 1_1 1_2$	$a_1 1_1 0_6$	$a_6 0_4 2_6$
$a_2 0_4 0_5$	$a_7 0_2 0_3$	$a_2 2_0 0_2$	$a_7 0_5 1_0$
$a_3 1_5 1_6$	$2_2 1_3 2_6$;	$a_3 2_1 0_3$	$0_1 1_4 1_6$
$a_4 0_6 0_0$	$2_0 1_4$	$a_4 1_2 2_4$	$0_0 1_3$
$a_5 1_0 2_1$	$0_1 2_5$	$a_5 2_3 1_5$	$2_2 2_5$

$a_1 1_2 0_5$	$a_6 0_0 0_3$	$a_1 2_0$	$a_6 2_5$	$1_1 1_3$
$a_2 2_3 1_6$	$a_7 0_1 2_4$	$a_2 2_1$	$a_7 2_6$	$2_2 0_5$
$a_3 0_4 2_0$	$1_0 1_4 2_5$;	$a_3 1_2$	$0_0 0_2$	$1_4 2_4$
$a_4 1_5 1_1$	$2_1 1_3$	$a_4 2_3$	$1_0 0_3$	$0_6 1_6$
$a_5 0_6 2_2$	$0_2 2_6$	$a_5 0_4$	$0_1 1_5$	

Four more parallel classes are:

$0_0 1_0 2_0$	$0_0 0_5 1_6$	$1_0 0_1 0_6$	$1_0 1_1 0_2$	
$0_1 1_2 0_4$	$0_1 1_1 2_1$	$0_2 1_2 2_2$	$0_3 1_3 2_3$	
$0_3 1_5 0_6$;	$0_2 1_3 0_4$;	$0_3 1_4 0_5$;	$0_4 1_6$	
$a_1 a_2 a_5$	$a_1 a_3 a_6$	$a_1 a_4 a_7$	$0_5 1_5 2_5$	mod (3, –)
$a_3 a_7$	$a_2 a_7$	$a_2 a_6$	$a_2 a_3 a_4$	
$a_4 a_6$	$a_4 a_5$	$a_3 a_5$	$a_5 a_6 a_7$	
			a_1	

Note that this design has a sub-RRP(4, 3) on the points a_7, 0_6, 1_6, 2_6. Delete the blocks in this subdesign.

R*RP(28, 17)-RRP(4, 3). Take a resolvable TD(3, 10) and remove two points from one of the groups. Apply Construction 2.2, filling in R*RP(10, 7) and an RRP(8, 7) – RRP(4, 3). The following R*RP(10, 7) is given in [20].

1, 5	1, 2	1, 7	1, 3, 8	2, 3, 9	1, 9, 10	6, 7, 8
2, 10	3, 4	2, 8	4, 5, 9	5, 7, 10	2, 4, 6	1, 4
3, 6	5, 6	3, 5	2, 7	1, 6	3, 7	2, 5
4, 7	7, 9	4, 10	6, 10	4, 8	5, 8	3, 10
8, 9	8, 10	6, 9				9

R*RP(28, 25)-RRP(4, 3). Our point set is $\{a_1, \ldots, a_{10}\} \cup \mathbb{Z}_{18}$.

	$a_1 0$	$a_6 8$	11, 16	
	$a_2 2$	$a_7 1$	3, 7	
Eighteen parallel classes:	$a_3 4$	$a_8 13$	10, 17	mod 18
	$a_4 5$	$a_9 14$	12, 15	
	$a_5 6$	$a_{10} 9$		

Seven classes on $\mathbb{Z}_{18}$ are obtained as follows. Three classes of pairs are given by developing the class $0,9,\ \ 6,15,\ \ 3,12,\ \ 8,16,\ \ 7,17,\ \ 4,14,\ \ 5,13,\ \ 2,10,\ \ 1,11 \bmod 18$; a class of triples arises from developing the block $0,6,12 \bmod 18$. The remaining three classes are:

(i) $0+i, 1+i, 2+i \quad i=0, 3, 6, 9, 12, 15$

(ii) $1+i, 3+i$
$2+i, 4+i \quad i=0, 6, 12$
$5+i, 6+i$

(iii) $1+i, 3+i$
$2+i, 4+i \quad i=3, 9, 15.$
$5+i, 6+i$

Now construct an R*RP(10, 7) (see previous design) on $a_1, \ldots, a_{10}$ so that $\{a_1, a_{10}\}$ is a block of size two. Then our R*RP(28, 25) has a sub-RRP(4, 3) on the points a_1, a_{10}, 0, 9; delete the blocks in this subdesign.

R*RP(28, 26)-RRP(4, 3). Our point set is $\{a_1, \ldots, a_{10}\} \cup \mathbb{Z}_{18}$.
Proceed as above except that the last three classes on $\mathbb{Z}_{18}$ are replaced by the four classes

$0+i, 1+i$	$2+i, 4+i$	$9+i, 11+i$	
$6+i, 7+i$	$3+i, 5+i$	$14+i, 16+i$	$i=0, 2, 4$
$12+i, 13+i$	$8+i, 10+i$	$15+i, 17+i$	

and $1+i, 2+i, i=0, 2, 4, \ldots, 16$.

On the points $a_1, \ldots, a_{10}$ construct an R*RP(10, 8) (the following such design can be found in [20]):

0, 9	1, 9	2, 9	3, 9	4, 9	0, 1, 2	0, 3, 4	0
4, 5	0, 5	0, 6	0, 7	0, 8	5, 6, 9	7, 8, 9	9
3, 6	2, 8	1, 3	2, 4	1, 7	3, 8	2, 5	2, 3
2, 7	3, 7	4, 8	1, 5	2, 6	4, 7	1, 6	1, 4
1, 8	4, 6	5, 7	6, 8	3, 5			5, 8
							6, 7

Again by constructing this so that $\{a_1, a_{10}\}$ is a block of size two our R*RP(28, 26) will have a sub-RRP(4, 3) on the points a_1, a_{10}, 0, 9 and we then just delete the blocks in this subdesign.

A RESOLVABLE 4-GDD of type 3^8.

Groups:				
	1, 5, 9	2, 6, 10	3, 7, 11	4, 8, 12
	13, 17, 21	14, 18, 22	15, 19, 23	16, 20, 24

Blocks:

1, 6, 15, 16	1, 2, 11, 18	1, 8, 22, 23
2, 9, 23, 24	3, 10, 13, 24	2, 3, 12, 19
3, 8, 17, 18	4, 6, 7, 23	4, 15, 17, 24
4, 11, 13, 14	5, 12, 14, 15	5, 13, 16, 18
5, 10, 19, 20	8, 16, 19, 21	6, 11, 20, 21
7, 12, 21, 22	9, 17, 20, 22	7, 9, 10, 14

1, 14, 21, 24	1, 3, 4, 20	1, 7, 13, 19	1, 10, 12, 17
2, 7, 16, 17	2, 13, 15, 22	2, 8, 14, 20	2, 4, 5, 21
3, 5, 6, 22	5, 7, 8, 24	3, 9, 15, 21	3, 14, 16, 23
4, 9, 18, 19	6, 14, 17, 19	4, 10, 16, 22	6, 8, 9, 13
8, 10, 11, 15	9, 11, 12, 16	5, 11, 17, 23	7, 15, 18, 20
12, 13, 20, 23	10, 18, 21, 23	6, 12, 18, 24	11, 19, 22, 24